Malaysian Journal of Mathematical Sciences 10(S) February: 55–60 (2016)
Special Issue: The 3^{rd} International Conference on Mathematical Applications in Engineering 2014 (ICMAE'14)

MALAYSIAN JOURNAL OF MATHEMATICAL SCIENCES

Journal homepage: http://einspem.upm.edu.my/journal

On the Uniform Convergence of Fourier Series.

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ABSTRACT

We prove theorems on the uniform summabilty of Fourier-Laplace series of distributions from the Sobolev spaces with negative smoothness defined on a unique sphere.

Keywords: Fourier-Laplace series, Summability, Localization, Chezaro Means.

1. Introduction

Let B^{N+1} the unit ball in R^{N+1}. Denote its surface as:

$$S^N = \{x = (x_1, x_2, ..., X_{N+1}) \in R^{N+1} : \sum_{n=1}^{N+1} x_n^2 = 1\}$$

Let x and y in S^N . By $\gamma = \gamma(x, y)$ denote spherical distance between these two points. In fact $\gamma \leq \pi$ is the angle between vectors x and y. Denote $B(x, r)$ a geodesic ball on S^N of radius r centred at x:

$$B(x, r) = \{y \in S^N : \gamma(x, y) \leq r\}$$

The Laplace-Beltrami operator Δ_s on S^N can be defined using the Laplace Δ in R^{N+1} (see for instance in Shubin (2007).): let $f(x)$ a function defined on S^N. Then extend it to R^{N+1}, by putting $\hat{f}(x) = f\left(\frac{x}{|x|}\right)$, $x \in R^{N+1}$. Then $\Delta_s f = \Delta \hat{f}\big|_{S^N}$. The Laplace-Beltrami operator can be also derived from the space Laplace operator presenting it in spherical coordinates:

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{N}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_s,$$

where the Laplace-Beltrami operator Δ_s can be written in spherical coordinates $(\xi_1, \xi_2, ..., \xi_{N-1}, \zeta)$ as:

$$\begin{aligned}\Delta_s = & \frac{1}{\sin^{N-1}\xi_1}\frac{\partial}{\partial \xi_1}\left(\sin^{N-1}\xi_1 \frac{\partial}{\partial \xi_1}\right) + \\ & \frac{1}{\sin^2\xi_1 \sin^{N-2}\xi_2}\frac{\partial}{\partial \xi_2}\left(\sin^{N-2}\xi_2 \frac{\partial}{\partial \xi_2}\right) + \cdots \\ & + \frac{1}{\sin^2\xi_1 \sin^2\xi_2 \; . \; . \; . \sin^2\xi_{N-1}}\frac{\partial^2}{\partial \zeta^2}.\end{aligned}$$

Operator $-\Delta_s$ as a formal differential operator with domain of definition $C^\infty\left(S^N\right)$ is a symmetric, non negative and its closure $\overline{-\Delta_s}$ is a selfadjoint operator in $L_2\left(S^N\right)$. Thus, it is essentially sefadjoint operator in the Hilbert space $L_2(S^N)$. The eigenfunctions Y^k of this operator are called spherical harmonics. The spherical harmonics are mutually orthogonal and associated with eigenvalues $\lambda_k = k(k+N-1),\ k = 0, 1, 2, ...$, with the frequency $a_k = N_k - N_{k-2}$, where $N_k = \frac{(N+k)!}{N!k!}$. That is why for each k there are a_k number of spherical harmonics $\left\{Y_j^k\right\}\Big|_{j=1}^{a_k}$ corresponding to eigenvalue λ_k . A family of functions $\left\{Y_j^k\right\}\Big|_{j=1}^{a_k}$ is an orthonormal basis in the space of spherical harmonics of a degree k which we denote by $\aleph_k$.

Note that an arbitrary function $f \in L_2\left(S^N\right)$ can be represented in a unique way as Fourier series by spherical harmonics $\left\{Y_j^k\right\}\Big|_{j=1}^{a_k}$. Such a series is called Fourier-Laplace series on sphere:

$$f(x) = \sum_{k=0}^{\infty}\sum_{j=1}^{a_k} f_{k,j} Y_j^k(x), \tag{1}$$

where $f_{k,j} = \int_{S^N} f(y) Y_j^k(y) d\sigma(y)$, and equality (1.1) should be understanding in sense of $L_2\left(S^N\right)$.

Let denote by $S_n f(x)$ a partial sum of series (1.1). It is clear that in $S_n f(x)$ by changing order of integration and summation one can easily rewrite it as:

$$S_n f(x) = \int_{S^N} f(y)\Theta(x,y,n)d\sigma(y),$$

where a function $\Theta(x,y,n)$ is a spectral function (see in Alimov (1993)) of a selfadjoint operator $\overline{-\Delta}$ and has a form:

$$\Theta(x,y,n) = \sum_{k=0}^{n}\sum_{j=1}^{a_k} Y_j^k(x)Y_j^k(y), \tag{2}$$

and $S_n f(x)$ is called a spectral expansion of an element f corresponding to the operator $\overline{-\Delta}$ (see in Alimov (1993)).

2. Preliminaries

1Â°. Determine Chezaro means of order δ of partial sums of series (1.1) by equality

$$S_n^\delta f(x) = \frac{1}{A_n^\delta}\sum_{k=0}^{n} A_{n-k}^\delta \sum_{j=1}^{a_k} f_{k,j}Y_j^k(x), \tag{3}$$

where $A_n^\delta = \frac{\Gamma(\delta+m+1)}{\Gamma(\delta+1)m!}$.

Definition 2.1. Series (1.1) is sumable to $f(x)$ by Chezaro means of order δ if it is true that

$$\lim_{n\to\infty} S_n^\delta f(x) = f(x) \tag{4}$$

In the determination equality (2.2) can be understood in any sense (topology). Here in the present article we will consider it in sense of almost every where convergence.

Note that Chezaro means of zero order is coincides with a partial sum $S_n f(x)$ and as a partial sum it also can be represented as an integral operator with a kernel which is Chezaro means of the spectral function (1.2)

$$\Theta^\delta(x,y,n) = \frac{1}{A_n^\delta}\sum_{k=0}^{n} A_{n-k}^\delta \sum_{j=1}^{a_k} Y_j^k(x)Y_j^k(y), \tag{5}$$

Thus formula (2.1) can be written as

$$S_n^\delta f(x) = \int_{S^N} f(y)\Theta^\delta(x,y,n)d\sigma(y). \tag{6}$$

By $P_k^\nu(t)$ denote Gegenbaur's polinomials (when $\nu = \frac{1}{2}$ the Legender polynomials) . Let's put $\nu = \frac{N-1}{2}$. A function $\Theta^\delta(x,y,n)$ can be represented in a form (see for instance in Pulatov (1981)) :

$$\Theta^\delta(x,y,n) = \frac{\Gamma(n+1)}{\Gamma(n+\delta+1)} \sum_{k=0}^{n} \frac{\Gamma(n-k+\delta+1)}{\Gamma(n-k+1)} \left(k+\nu\right)^\nu P_k^\nu(\cos\gamma). \tag{7}$$

The Nikol'skii class of functions on the sphere S^N are defined by passing to local coordinates from the definition of the Nikol'skii class in R^{N+1} (see. in Il'in (1991)). In Pulatov (1981) it is studied summability and localization conditions for the expansions (3) of functions in $H_p^\alpha(S^N)$ and proved following

Theorem 2.1. *Let $f \in H_p^\alpha(S^N)$, where $1 \le p \le \infty, \alpha > 0$ and $\delta > -1$. Then the following assertions are true*

1) If $p\alpha > N$ and $\alpha + \delta > \frac{N-1}{2}$, then

$$\lim_{n\to\infty} S_n^\delta f(x) = f(x)$$

uniformly on S^N.

2) If $\alpha + \delta > max\{\frac{N}{p} - 1, \frac{N-1}{2}\}$ and $f = 0$ in some domain $U \subset S^N$, then

$$\lim_{n\to\infty} S_n^\delta f(x) = 0$$

uniformly on each compact set $K \subset U$.

3) If $\alpha + \delta > \frac{N-1}{2}$ and $f = 0$ in some domain $U \subset S^N$ as well as diametrically opposite U, then

$$\lim_{n\to\infty} S_n^\delta f(x) = 0$$

uniformly on each compact set $K \subset U$.

In Pulatov (1981) it is also proved that condition $\alpha+\delta > \frac{N-1}{2}$ in the all three assertions of the Theorem 2.1 cannot be replaced by the condition $\alpha+\delta = \frac{N-1}{2}$ and also shown that localization is not guaranteed by the conditions $p\alpha > N, \delta > -1, \alpha + \delta = \frac{N-p}{p}$.

Note that the three conditions $1 \le p \le \infty, p\alpha > N, \alpha+\delta > \frac{N-1}{2}$ for uniform convergence were fist obtained by Il'in for the Riesz means ($0 \le s < \frac{N-1}{2}$) of an expansions in eigenfunctions of Laplace operator Il'in (1991).

Let $C^\infty(S^N)$ is the space of infinitely differentiable on S^N functions. Then from Theorem 2.1 it follows

Proposition 1. *Let $f(x) \in C^\infty(S^N)$. Then Fourier-Laplace series (1) of this function and its partial derivatives of any order uniformly converge on S^N .*

Denote $D'(S^N)$ the space of all linear continuous functionals on $C^\infty(S^N)$. Elements of $D'(S^N)$ also known as distributions. For any distribution $\psi \in D'(S^N)$ and any test function $f \in C^\infty(S^N)$ by $< \psi, f >$ denote value of the functional ψ on $f(x)$. Since $Y_j^k(x) \in C^\infty(S^N)$, then define Fourier-Laplace coefficients of ψ as $< \psi, Y_j^k >$.

Definition 1. *We say sequence ψ_n from $D'(S^N)$ converges to $\psi \in D'(S^N)$ if for any function $f(x) \in C^\infty(S^N)$*

$$\lim_{n \to infty} < \psi_n, f > \quad = \quad < \psi, f > .$$

Convergence in Definition 1 recall convergence in $D'(S^N)$ topology. From Proposition 1 and Definition 1 it follows

Proposition 2. *For any distribution ψ its Fourier-Laplace series converges to itself in $D'(S^N)$ topology.*

Both propositions 1 and 2 first time proved in Alimov (1993) for the Fourier series in eigenfunction expansions associated with Laplace operator. Further in Alimov (1993) proved for the spectral expansions associated with general elliptic operators. These statements for the Fourier-Laplace series can be found in Rakhimov (1994).

Uniformly convergence of Fourier-Laplace series of a distribution ψ can be studied in the domains where a distribution coincides with a smooth function. For example in the domain where a distribution is equal zero (a localization problem).

Main Result

Theorem 2.2. *Let $f \in H_p^{-\alpha}(S^N)$, where $1 \le p \le \infty, \alpha > 0$ and $\delta > -1$.*

If $\delta > max\{\frac{N}{p} - 1, \frac{N-1}{2}\} + \alpha$ *and* $f = 0$ *in some domain* $U \subset S^N$*, then*

$$\lim_{n \to \infty} S_n^{\delta} f(x) = 0$$

uniformly on each compact set $K \subset U$*.*

In the case $p = 2$ this this theorem proved in Rakhimov (1994). Also in Rakhimov (1994) it is proved part 3 of the theorem 2.1 in the Sobolev spaces with negative smoothness. The Reisz means of the Fourier-Laplace series of distributions studied in Rakhimov (2013). Questions of almost everywhere summability of Fourier-Laplace series are studied in Rakhimov (2009).

Acknowledgments

Research on the topics of the paper supported by MOHE grant FRGS 14 142 0383 IIUM.

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